

# Regular graphs are antimagic

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May 1, 2015

## Abstract

In this note we prove - with a slight modification of an argument of Cranston et al. [2] - that  $k$ -regular graphs are antimagic for  $k \geq 2$ .

## 1 Introduction

Throughout the note graphs are assumed to be simple. Given an undirected graph  $G = (V, E)$  and a subset of edges  $F \subseteq E$ ,  $F(v)$  denotes the set of edges in  $F$  incident to node  $v \in V$ , and  $d_F(v) := |F(v)|$  is the **degree** of  $v$  in  $F$ . A **labeling** is an injective function  $f : E \rightarrow \{1, 2, \dots, |E|\}$ . Given a labeling  $f$  and a subset of edges  $F$ , let  $f(F) = \sum_{e \in F} f(e)$ . A labeling is **antimagic** if  $f(E(u)) \neq f(E(v))$  for any pair of different nodes  $u, v \in V$ . A graph is said to be **antimagic** if it admits an antimagic labeling.

Hartsfield and Ringel conjectured [4] that all connected graphs on at least 3 nodes are antimagic. The conjecture has been verified for several classes of graphs (see e.g. [3]), but is widely open in general. In [2] Cranston et al. proved that every  $k$ -regular graph is antimagic if  $k \geq 3$  is odd. Note that 1-regular graphs are trivially not antimagic. We have observed that a slight modification of their argument also works for even regular graphs, hence we prove the following.

**Theorem 1.** *For  $k \geq 2$ , every  $k$ -regular graph is antimagic.*

It is worth mentioning the following conjecture of Liang [5]. Let  $G = (S, T; E)$  be a bipartite graph. A path  $P = \{uv, vw\}$  of length 2 with  $u, w \in S$  is called an  **$S$ -link**.

**Conjecture 2.** *Let  $G = (S, T; E)$  be a bipartite graph such that each node in  $S$  has degree at most 4 and each node in  $T$  has degree at most 3. Then  $G$  has a matching  $M$  and a family  $\mathcal{P}$  of node-disjoint  $S$ -links such that every node  $v \in T$  of degree 3 is incident to an edge in  $M \cup (\bigcup_{P \in \mathcal{P}} P)$ .*

Liang showed that if the conjecture holds then it implies that every 4-regular graph is antimagic. The starting point of our investigations was proving Conjecture 2. As Theorem 1 provides a more general result, we leave the proof of Conjecture 2 for a forthcoming paper [1].

## 2 Proof of Theorem 1

A **trail** in a graph  $G = (V, E)$  is an alternating sequence of nodes and edges  $v_0, e_1, v_1, \dots, e_t, v_t$  such that  $e_i$  is an edge connecting  $v_{i-1}$  and  $v_i$  for  $i = 1, 2, \dots, t$ , and the edges are all distinct (but there might be repetitions among the nodes). The trail is **open** if  $v_0 \neq v_t$ , and **closed** otherwise. The **length** of a trail is the number of edges in it. A closed trail containing every edge of the graph is called an **Eulerian trail**. It is well known that a graph has an Eulerian trail if and only if it is connected and every node has even degree.

**Lemma 3.** *Given a connected graph  $G = (V, E)$ , let  $T = \{v \in V : d_E(v) \text{ is odd}\}$ . If  $T \neq \emptyset$ , then  $E$  can be partitioned into  $|T|/2$  open trails.*

*Proof.* Note that  $|T|$  is even. Arrange the nodes of  $T$  into pairs in an arbitrary manner and add a new edge between the members of every pair. Take an Eulerian trail of the resulting graph and delete the new edges to get the  $|T|/2$  open trails.  $\square$

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The main advantage of Lemma 3 is that the edge set of the graph can be partitioned into open trails such that at most one trail starts at every node of  $V$ . Indeed, there is a trail starting at  $v$  if and only if  $v$  has odd degree in  $G$ . This is how we see the Helpful Lemma of [2].

**Corollary 4** (Helpful Lemma of [2]). *Given a bipartite graph  $G = (U, W; E)$  with no isolated nodes in  $U$ ,  $E$  can be partitioned into subsets  $E^\sigma, T_1, T_2, \dots, T_l$  such that  $d_{E^\sigma}(u) = 1$  for every  $u \in U$ ,  $T_i$  is an open trail for every  $i = 1, 2, \dots, l$ , and the endpoints of  $T_i$  and  $T_j$  are different for every  $i \neq j$ .*

*Proof.* Take an arbitrary  $E' \subseteq E$  with the property  $d_{E'}(u) = 1$  for every  $u \in U$ . A component of  $G - E'$  containing more than one node is called **nontrivial**. If there exists a nontrivial component of  $G - E'$  that only contains even degree nodes then let  $uw_1 \in E - E'$  be an edge in this component with  $u \in U$  and  $w_1 \in W$ , and let  $uw_2 \in E'$ . Replace  $uw_2$  with  $uw_1$  in  $E'$ . After this modification, the component of  $G - E'$  that contains  $u$  has an odd degree node, namely  $w_1$ . Iterate this step until every nontrivial component of  $G - E'$  has some odd degree nodes. Let  $E^\sigma = E'$  and apply Lemma 3 to get the decomposition of  $E - E^\sigma$  into open trails.  $\square$

In what follows we prove that regular graphs are antimagic: for sake of completeness we include the odd regular case, too. We emphasize the differences from the proof appearing in [2].

*Proof of Theorem 1.* Note that it suffices to prove the theorem for connected regular graphs. Let  $G = (V, E)$  be a connected  $k$ -regular graph and let  $v^* \in V$  be an arbitrary node. Denote the set of nodes at distance exactly  $i$  from  $v^*$  by  $V_i$  and let  $q$  denote the largest distance from  $v^*$ . We denote the edge-set of  $G[V_i]$  by  $E_i$ . Apply Corollary 4 to the induced bipartite graph  $G[V_{i-1}, V_i]$  with  $U = V_i$  to get  $E_i^\sigma$  and the trail decomposition of  $G[V_{i-1}, V_i] - E_i^\sigma$  for every  $i = 1, \dots, q$ . The edge set of  $G[V_{i-1}, V_i] - E_i^\sigma$  is denoted by  $E'_i$ .

Now we define the antimagic labeling  $f$  of  $G$  as follows. We reserve the  $|E_q|$  smallest labels for labeling  $E_q$ , the next  $|E_q^\sigma|$  smallest labels for labeling  $E_q^\sigma$ , the next  $|E'_q|$  smallest labels for labeling  $E'_q$ , the next  $|E_{q-1}|$  smallest labels for labeling  $E_{q-1}$ , etc. There is an important difference here between our approach and that of [2] as we switched the order of labeling  $E_i^\sigma$  and  $E'_i$ , and we don't yet define the labels, we only reserve the intervals to label the edge sets. Next we prove a claim that tells us how to label the edges in  $E'_i$ .

**Claim 5.** *Assume that we have to label the edges of  $E'_i$  from interval  $s, s+1, \dots, \ell$  (where  $|E'_i| = \ell - s + 1$ ), and that we are given a trail decomposition of  $E'_i$  into open trails. We can label  $E'_i$  so that successive labels (in a trail) incident to a node  $v_i \in V_i$  have sum at most  $s + \ell$ , and successive labels (in a trail) incident to a node  $v_{i-1} \in V_{i-1}$  have sum at least  $s + \ell$ .*

*Proof.* Our proof of this claim is essentially the same as the proof in [2]: we merely restate it for self-containedness. Let  $\mathcal{T}$  be the trail decomposition of  $E'_i$  into open trails. Take an arbitrary trail  $T = u_0, e_1, u_1, \dots, e_t, u_t$  of length  $t$  from  $\mathcal{T}$  and consider the following two cases (see Figure 1 for an illustration).

- **Case A:** If  $u_0 \in V_{i-1}$  then label  $e_1, \dots, e_t$  by  $s, \ell, s+1, \ell-1, \dots$  in this order. In this case the sum of 2 successive labels is  $s + \ell$  at a node in  $V_i$ , and it is  $s + \ell + 1$  at a node in  $V_{i-1}$ .
- **Case B:** If  $u_0 \in V_i$  then label  $e_1, \dots, e_t$  by  $\ell, s, \ell-1, s+1, \dots$  in this order. In this case the sum of 2 successive labels is  $s + \ell - 1$  at a node in  $V_i$ , and it is  $s + \ell$  at a node in  $V_{i-1}$ .

We prove by induction on  $|\mathcal{T}|$ . The proof is finished by the following cases.

1. If  $\mathcal{T}$  contains a trail of even length, then let  $T$  be such a trail (and again  $t$  denotes the length of  $T$ ). If the endpoints of  $T$  fall in  $V_{i-1}$  then apply Case A. On the other hand, if the endpoints of  $T$  fall in  $V_i$  then apply Case B. In both cases we use  $\frac{t}{2}$  labels from the lower end of the interval, and  $\frac{t}{2}$  labels from the upper end, therefore we can label the edges of the trails in  $\mathcal{T} - T$  from the (remaining) interval  $s + \frac{t}{2}, s + \frac{t}{2} + 1, \dots, \ell - \frac{t}{2}$ , so that the lower bound  $s + \frac{t}{2} + \ell - \frac{t}{2} = s + \ell$  holds for the sum of two successive labels at every  $v_{i-1} \in V_{i-1}$ , and the same upper bound holds at each node  $v_i \in V_i$ .
2. Every trail in  $\mathcal{T}$  has odd length. If  $\mathcal{T}$  contains only one trail then label it using either of the two cases above and we are done. Otherwise let  $T_1$  and  $T_2$  be two trails from  $\mathcal{T}$ , and let  $t_i$  be the length of  $T_i$  for both  $i = 1, 2$ . Label first the edges of  $T_1$  using Case A (starting at the endpoint of  $T_1$  that lies in  $V_{i-1}$ ). Note that the remaining labels form the interval  $s + \frac{t_1+1}{2}, \dots, \ell - \frac{t_1-1}{2}$ . Next label the edges of  $T_2$  using Case B (starting at the endpoint of  $T_2$  that lies in  $V_i$ ). Note that the sum of successive labels in the trail  $T_2$  becomes  $s + \frac{t_1+1}{2} + (\ell - \frac{t_1-1}{2}) - 1 = s + \ell$  at a node in  $V_i$ , and it is  $s + \frac{t_1+1}{2} + (\ell - \frac{t_1-1}{2}) = s + \ell + 1$  at a node in  $V_{i-1}$ , which is fine for us. Finally, the remaining labels form the interval  $s + \frac{t_1+1}{2} + \frac{t_2-1}{2}, \dots, \ell - \frac{t_1-1}{2} - \frac{t_2+1}{2}$ , therefore we can label the edges of the trails in  $\mathcal{T} - \{T_1, T_2\}$  from the remaining interval so that the lower bound  $s + \frac{t_1+1}{2} + \frac{t_2-1}{2} + \ell - \frac{t_1-1}{2} - \frac{t_2+1}{2} = s + \ell$  holds for the sum of two successive labels at every node of  $V_{i-1}$ , and the same upper bound holds at every node of  $V_i$ .  $\square$

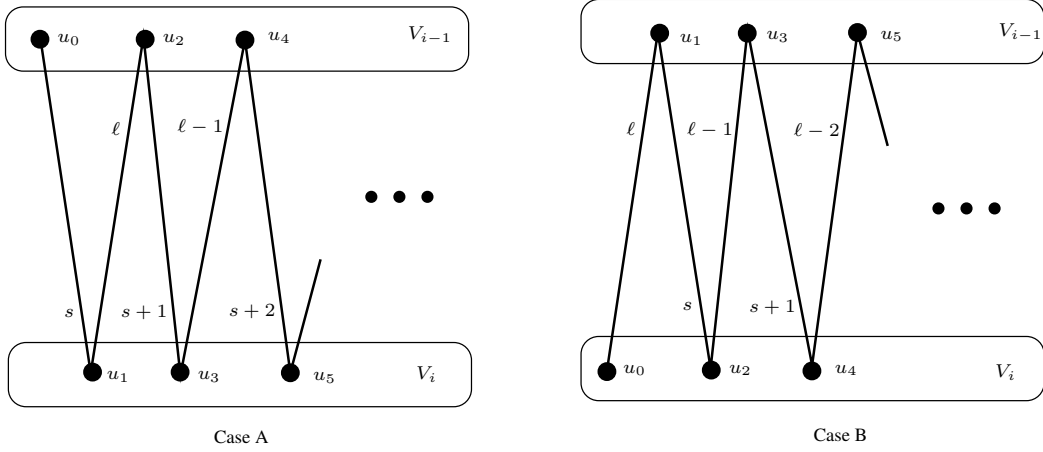


Figure 1: An illustration for labeling trails.

Now we specify how the labels are determined to make sure  $f(E(u)) \neq f(E(v))$  for every  $u \neq v$ . We label the edges of every  $E_i$  arbitrarily from their dedicated intervals. Label the edges of every  $E'_i$  in the manner described by Claim 5. For any node  $v \in V_i$  with  $i > 0$ , let  $\sigma(v)$  denote the unique edge of  $E'_i$  incident to  $v$ . Let  $p(v) = f(E(v)) - f(\sigma(v))$  for every  $v \in V - v^*$ . We label the edges in  $E_q^\sigma, E_{q-1}^\sigma, \dots, E_1^\sigma$  as in [2]: if we already labeled  $E_q^\sigma, E_{q-1}^\sigma, \dots, E_{i+1}^\sigma$  then  $p(v_i)$  is already determined for every  $v_i \in V_i$ . So we order the nodes of  $V_i$  in an increasing order according to their  $p$ -value and assign the label to their  $\sigma$  edge in this order. This ensures that  $f(E(u)) \neq f(E(v))$  for an arbitrary pair  $u, v \in V_i$ .

We have fully described the labeling procedure. This labeling scheme ensures that  $f(E(v_i)) < f(E(v_j))$  if  $v_i \in V_i, v_j \in V_j$  and  $i \geq j + 2$  since  $G$  is regular and the edges in  $E(v_j)$  get larger labels than those in  $E(v_i)$ . Similarly,  $f(E(v^*)) > f(E(v))$  for every  $v \in V - v^*$  for the same reason. It is only left is to show that  $f(E(v_i)) \neq f(E(v_{i-1}))$  for arbitrary  $v_i \in V_i, v_{i-1} \in V_{i-1}$  and  $i \geq 2$ .

**Claim 6.** For arbitrary  $v_i \in V_i, v_{i-1} \in V_{i-1}$  and  $i \geq 2$  we have

$$(i) \ p(v_i) \leq \frac{k-2}{2}(s+\ell) + \ell \text{ and } p(v_{i-1}) \geq \frac{k-2}{2}(s+\ell) + s, \text{ if } k \text{ is even, and}$$

$$(ii) \ p(v_i) \leq \frac{k-1}{2}(s+\ell) \text{ and } p(v_{i-1}) \geq \frac{k-1}{2}(s+\ell), \text{ if } k \text{ is odd.}$$

*Proof.* Assume first that  $k$  is even. In this case  $p(v)$  is the sum of an odd number of labels. We pair up all but one of these labels using the trail decomposition of  $E'_i$  to get the bounds needed.

1. Take a node  $v_i \in V_i$ . Note that  $f(e) < s$  for every  $e \in E(v_i) - E'_i$ . Let  $t = d_{E'_i}(v_i)$ .
  - (a) If  $t$  is even then  $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t}{2}(s+\ell)$  by Claim 5, giving  $p(v_i) \leq \frac{t}{2}(s+\ell) + (k-1-t)s \leq \frac{k-2}{2}(s+\ell) + \ell$ .
  - (b) If  $t$  is odd then  $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t-1}{2}(s+\ell) + \ell$  by Claim 5, giving  $p(v_i) \leq \frac{t-1}{2}(s+\ell) + \ell + (k-1-t)s \leq \frac{k-2}{2}(s+\ell) + \ell$ .
2. Now take a node  $v_{i-1} \in V_{i-1}$ . Note that  $f(e) > \ell$  for every  $e \in E(v_{i-1}) - E'_i$ . Let again  $t = d_{E'_i}(v_{i-1})$ .
  - (a) If  $t$  is even then  $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t}{2}(s+\ell)$  by Claim 5, giving  $p(v_{i-1}) \geq \frac{t}{2}(s+\ell) + (k-1-t)\ell \geq \frac{k-2}{2}(s+\ell) + s$ .
  - (b) If  $t$  is odd then  $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t-1}{2}(s+\ell) + s$  by Claim 5, giving  $p(v_{i-1}) \geq \frac{t-1}{2}(s+\ell) + s + (k-1-t)\ell \geq \frac{k-2}{2}(s+\ell) + s$ .

This concludes the proof of (i).

Although the proof of (ii) can be found in [2], we also present it here to make the paper self contained. The proof is very similar to the even case. So assume that  $k$  is odd. In this case  $p(v)$  is the sum of an even number of labels. We pair up these labels using the trail decomposition of  $E'_i$  to get the bounds needed.

1. Take a node  $v_i \in V_i$ . Note that  $f(e) < s$  for every  $e \in E(v_i) - E'_i$ . Let  $t = d_{E'_i}(v_i)$ .
  - (a) If  $t$  is even then  $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t}{2}(s+\ell)$  by Claim 5, giving  $p(v_i) \leq \frac{t}{2}(s+\ell) + (k-1-t)s \leq \frac{k-1}{2}(s+\ell)$ .

- (b) If  $t$  is odd then  $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t-1}{2}(s+\ell) + \ell$  by Claim 5, giving  $p(v_i) \leq \frac{t-1}{2}(s+\ell) + \ell + (k-1-t)s \leq \frac{k-1}{2}(s+\ell)$ .
2. Now take a node  $v_{i-1} \in V_{i-1}$ . Note that  $f(e) > \ell$  for every  $e \in E(v_{i-1}) - E'_i$ . Let again  $t = d_{E'_i}(v_{i-1})$ .
- (a) If  $t$  is even then  $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t}{2}(s+\ell)$  by Claim 5, giving  $p(v_{i-1}) \geq \frac{t}{2}(s+\ell) + (k-1-t)\ell \geq \frac{k-1}{2}(s+\ell)$ .
- (b) If  $t$  is odd then  $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t-1}{2}(s+\ell) + s$  by Claim 5, giving  $p(v_{i-1}) \geq \frac{t-1}{2}(s+\ell) + s + (k-1-t)\ell \geq \frac{k-1}{2}(s+\ell)$ .

This concludes the proof of (ii), and we are done.  $\square$

The assignment of the labels implies  $f(\sigma(v_i)) < s$  and  $f(\sigma(v_{i-1})) > \ell$  for  $v_i \in V_i$  and  $v_{i-1} \in V_{i-1}$ . Claim 6 yields  $f(E(v_i)) < f(E(v_{i-1}))$ , finishing the proof of Theorem 1.  $\square$

**Remark 7.** Observe that the proof of Theorem 1 does not really use the regularity of the graph, it merely relies on the fact that the degree of a node  $v_i \in V_i$  is not smaller than that of a node  $v_j \in V_j$  where  $i < j$ . Hence the following result immediately follows.

**Theorem 8.** Assume that a connected graph  $G = (V, E)$  ( $|V| \geq 3$ ) has a node  $v^* \in V$  of maximum degree such that  $d_E(v_i) \geq d_E(v_j)$  whenever  $v_i \in V_i, v_j \in V_j$  and  $i < j$ , where  $V_\ell$  denotes the set of nodes at distance exactly  $\ell$  from  $v^*$ . Then  $G$  is antimagic.

## Acknowledgement

The first and the second authors were supported by the Hungarian Scientific Research Fund - OTKA, K109240. The third author would like to thank Zhejiang Normal University, China - where he first heard about these problems - for their hospitality.

## References

- [1] K. Bérczi, A. Bernáth, and M. Vizer. A note on  $v$ -free 2-matchings. Manuscript, 2015.
- [2] D. W. Cranston, Y.-C. Liang, and X. Zhu. Regular graphs of odd degree are antimagic. *Journal of Graph Theory*, 2014.
- [3] J. A. Gallian. A dynamic survey of graph labeling. *The electronic journal of combinatorics*, 16(6):1–219, 2009.
- [4] N. Hartsfield and G. Ringel. Pearls in graph theory. 1990.
- [5] Y.-C. Liang. *Anti-magic labeling of graphs*. PhD thesis, National Sun Yat-sen University, 2013.